JOURNAL OF ALGEBRA **146**, 521–526 (1992)

On a Surmise of McAdam Concerning Quintasymptotic Primes

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Communicated by Melvin Hochster

Received June 18, 1990

S. McAdam has proved a result on the lifting and contraction of quintasymptotic primes under a certain assumption about finiteness, and has surmised that this assumption could be removed. Using the properties of excellent rings and Henselization, we extend his work and also give a generalized and simplified account of a result of Heitmann on which McAdam's result depends. © 1992 Academic Press, Inc.

1. INTRODUCTION

All rings in this paper are assumed to be commutative with an identity element.

In his monograph [8], McAdam surveyed recent developments in the theory of asymptotic primes and in particular dealt with the properties of so-called quintasymptotic primes. To describe these primes and the results referred to above, we recall some notation and terminology from [8].

The M-adic completion of a Noetherian local ring (A, M) will be denoted by (A^*, M^*). If R is a Noetherian ring with I an ideal of R, then the set $\bar{Q}^*(I)$ of quintasymptotic primes of I is defined as follows:

$$\bar{Q}^*(I) = \{P \in \operatorname{Spec} R \mid I \subseteq P,\ P_P^* \text{ minimal over } IR_P^* + q,\ \exists q \in \operatorname{Min}\operatorname{Spec} R_P^*\}.$$

McAdam goes on to prove the following result about their behaviour under lifting and contraction:

(cf. [8, (1.13)(b)]) *Let R and T be Noetherian rings with $R \subseteq T$ an integral extension. Suppose that T is a finite R-module. If $P \in \bar{Q}^*(I)$, then there is a $Q \in \bar{Q}^*(IT)$ with $Q \cap R = P$. If every minimal prime in T contracts to a minimal prime in R, then the converse holds.*

0021-8693/92 $3.00

(For another result of this type, in the presence of faithful flatness, see [8, (1.9)].) He goes on to surmise that the hypothesis that T be a finite R-module can be dispensed with, for a number of reasons (cf. [8, (1.14)]). His proof depends on the following result due to Heitmann:

(cf. [8, (1.12)]) *Let R be a domain, T a ring with $R \subseteq T$, and let $R^{\#}$ be a faithfully flat extension of R.*

(a) *Suppose that non-zero elements of R are regular in T. Then regular elements of $R^{\#}$ are regular in $R^{\#} \otimes_R T$.*

(b) *Suppose that each minimal prime in T contracts to 0 in R and that $R^{\#} \otimes_R T$ has only finitely many minimal primes. Then minimal primes in $R^{\#} \otimes_R T$ contract to minimal primes in $R^{\#}$.*

(Note that R need not be Noetherian in this result.)

The proof uses prime avoidance and careful choices of suitable elements.

For a detailed survey of the theory of various types of asymptotic primes, including quintasymptotic primes, we refer the reader to [8] (and [7, 4]). For basic facts about excellent rings, see [5, 6].

2. Preliminaries

We now collect together some facts about flatness, Hensel rings and Henselization which we shall need below.

For the basic properties of flatness and faithful flatness (and integral extensions) see [5, Chap. 2], for example. In particular we recall that if $A \to B$ is a flat homomorphism of rings, then $A \to B$ satisfies "Going-down," while if $A \to B$ is a faithfully flat homomorphism of rings, then it is injective.

For the properties of Hensel rings and Henselization that we require, we refer to [3, Sect. 18.6; 10; 9, Sects. 30, 43]. These properties are needed in the following very specific context.

Let (R, P) be a Noetherian local ring and let $R \subseteq T$ be an integral extension, where T is a Noetherian ring. Then T is a semilocal ring. Let $T' = R^* \otimes_R T$. Then $R^* \subseteq T'$, by flatness, and T' is integral over R^*, by base change. Again by base change, $T \subseteq T'$ is a faithfully flat extension. Let Q' be a maximal ideal of T'. Then

$$(Q' \cap T) \cap R = Q' \cap R = Q' \cap R^* \cap R = P^* \cap R = P;$$

hence $Q := Q' \cap T$ is a maximal ideal of T. Moreover

$$T'/QT' = R^* \otimes_R T/(\{x \otimes 1, 1 \otimes y \mid x \in P, y \in Q\}) = T/Q, \qquad \text{a field.}$$

Hence, $QT'=Q'$. In fact it therefore follows from faithful flatness that $QT' \in \operatorname{Max\,Spec} T'$, for each $Q \in \operatorname{Max\,Spec} T$. Thus we see in this way that T' is a semi-local integral extension of R^*. But R^* is Henselian [9, (30.3)], so T' is Henselian [3, (18.6.8)]. It follows from [3, (18.6.7)] and the above description of the maximal ideals of T' that the equation

$$T' = \Pi\{T'_Q \mid Q \in \operatorname{Max\,Spec} T\}$$

holds (where, in the expression T'_Q, T' is regarded as a T-module). This exhibits T' as a finite direct product of Henselian local rings T'_Q, $Q \in \operatorname{Max\,Spec} T$. In fact, by [10, p. 7, Proposition 2], each such T'_Q is integral over R^*. We deduce also that the QT'-adic completion of T' (which is the QT'_Q-adic completion of T'_Q) equals T^*_Q. Hence, we have the natural map $T'_Q \to T^*_Q$.

Now let $\tilde{T}_Q$ denote the Henselization of T_Q. Since T'_Q and the complete local ring T^*_Q are Henselian, the functorial properties of Henselization (cf. [10, p. 80], say) yield the following commutative diagram, with the obvious natural maps:

$$\begin{tikzcd} & \tilde{T}_Q \arrow[dr] \arrow[dd] & \\ T_Q \arrow[ur] \arrow[dr] & & T^*_Q. \\ & T'_Q \arrow[ur] & \end{tikzcd}$$

By base change, $T_Q \to T'_Q$ is faithfully flat, as is $T_Q \to \tilde{T}_Q$ (and $\tilde{T}_Q$ is local and Noetherian); moreover T^*_Q is also the completion of $\tilde{T}_Q$ at the maximal ideal so $\tilde{T}_Q \to T^*_Q$ is also faithfully flat (cf. [3, 18.6.6], for example). (In particular $\tilde{T}_Q \to T'_Q$ is injective, since $\tilde{T}_Q \to T^*_Q$ is.) It follows from [3, (18.6.8)] that $\tilde{T}_Q$ is isomorphic to $\tilde{R} \otimes_R T_Q$, where $\tilde{R}$ is the Henselization of R. Note that $\tilde{R}$ is a Noetherian local ring with completion R^* (see [3, (18.6.6)], say). Moreover, if $\tilde{T}$ denotes the Henselization of T, it follows from [3, (18.6.7) and (18.6.8)] that $\tilde{T}$ is isomorphic to $\tilde{R} \otimes_R T$, that the maximal ideals of $\tilde{T}$ are uniquely of the form $Q\tilde{T}$, for $Q \in \operatorname{Max\,Spec} T$, and that $(\tilde{T})_Q = \tilde{T}_Q$.

3. Results

We now present a generalization of Heitmann's result, using a more functorial approach. As usual, given a ring homomorphism $\varphi\colon A \to B$ and $P \in \operatorname{Spec} B$, $\varphi^{-1}(P)$ is written $P \cap A$ and is called the *contraction* of P to A;

moreover an element $a \in A$ is thought of as being in B, by considering $\varphi(a)$ acting on B.

PROPOSITION. *Let $R \to T$ and $R \to R^{\#}$ be ring homomorphisms, with the latter making $R^{\#}$ a flat R-module.*

(i) *Suppose that R is a domain and the non-zero elements of R are regular in T. Then the regular elements in $R^{\#}$ are regular in $R^{\#} \otimes_R T$.*

(ii) *Suppose that each minimal prime in T contracts to a minimal prime in R. Then a minimal prime in $R^{\#} \otimes_R T$ contracts to a minimal prime in $R^{\#}$.*

Proof. (i) Let K denote the quotient field of R and let $S = R \setminus \{0\}$. Denote $S^{-1}T$ by $\bar{T}$ and $S^{-1}R^{\#}$ by $\overline{R^{\#}}$. It easily follows from the hypotheses that $K \subseteq \bar{T}$, $K \subseteq \overline{R^{\#}}$, and $T \subseteq \bar{T}$. By base change, $\overline{R^{\#}} \subseteq \overline{R^{\#}} \otimes_K \bar{T}$ is a (faithfully) flat extension and $R^{\#} \subseteq \overline{R^{\#}}$ is a flat extension. Hence, $R^{\#} \subseteq \overline{R^{\#}} \otimes_K \bar{T}$ is a flat extension. But

$$\overline{R^{\#}} \otimes_K \bar{T} = R^{\#} \otimes_R T \otimes_R K = R^{\#} \otimes_R \bar{T} \supseteq R^{\#} \otimes_R T, \text{ since } R^{\#} \text{ is flat.}$$

The result follows.

(ii) By base change, $T \to R^{\#} \otimes_R T$ is a flat extension. Let Q be a minimal prime in the latter ring. Then $Q \cap T$ is minimal in T by Going-down. Hence, $Q \cap R =: P$ is a minimal prime in R. Apply the functor $- \otimes_R R/P$ and assume that the result holds in the case where R is a domain. It follows easily that $Q \cap R^{\#}$ is minimal over $PR^{\#}$. Since P is minimal in R, it is then clear that $Q \cap R^{\#}$ is in fact minimal in $R^{\#}$. So we reduce to the case where R is a domain, with quotient field K (say).

Apply the localization functor $K \otimes_R -$, denoting Q_S by $\bar{Q}$. Then $\bar{Q}$ is minimal in $R^{\#} \otimes_R T \otimes_R K$, where, as seen above, the latter equals $\overline{R^{\#}} \otimes_K \bar{T}$. Now, as before, $\overline{R^{\#}} \to \overline{R^{\#}} \otimes_K \bar{T}$ is (faithfully) flat. Hence, $\bar{Q} \cap \overline{R^{\#}}$ is minimal, by Going-down, so $Q \cap R^{\#}$ is also minimal as required.

We can now give our main results; our proofs are a mixture of McAdam's arguments (which we rely on implicitly for details and to which we refer the reader) and various aspects of the material in Section 2, which we use freely. We also recall the notation of Section 1; in particular $R \subseteq T$ is an integral extension of Noetherian rings with I an ideal of R, while if R is local, $T' = R^* \otimes_R T$.

We first extend McAdam's result on the lifting of quintasymptotic primes.

THEOREM 1. *Let $P \in \bar{Q}^*(I)$.*

(i) *If* $R_P^* \otimes_R T$ *is Noetherian, then there exists* $Q \in \bar{Q}^*(IT)$ *such that* $Q \cap R = P$.

(ii) *If* R_P *is excellent (and so if* R *is excellent), then there exists* $Q \in \bar{Q}^*(IT)$ *such that* $Q \cap R = P$.

Proof. (i) It may be assumed that R is local at P. Then there is a minimal prime q in R^* with P^* minimal over $IR^* + q$. By the properties of T', there is a minimal prime q' in T' and a (unique) maximal ideal P' in T' with $q' \subseteq P'$ such that $q' \cap R^* = q$, $P' \cap R^* = P^*$. Clearly P' is minimal over $IT' + q'$, so that $P'_{P'}$ is minimal over $IT'_{P'} + q'_{P'}$. Now $P' = QT'$ for a unique maximal ideal Q in T and $T'_{P'} = T'_Q$; moreover $T'_Q \to T_Q^*$ is faithfully flat, since T_Q^* is the completion of the Noetherian local ring T'_Q. By [2, Chapt. II, Sect. 2.6., Proposition 16] (or by faithful flatness) there is a minimal prime z in T_Q^* lying over $q'_{P'} = q'_Q$. Clearly Q^* is minimal over $IT_Q^* + z$, and the result follows.

(ii) As before, it may be assumed that R is local at P and that P^* is minimal over $IR^* + q$, where q is a minimal prime in R^*. Now R^* is the completion of the Henselization $\tilde{R}$ of R, so $w := q \cap \tilde{R}$ is a minimal prime in $\tilde{R}$, by Going-down. Now $\tilde{R}$ is excellent [3, (18.7.6)] so $\tilde{R}/w$ is excellent and Henselian (cf. [9, (43.4)], say). It follows from [3, (18.9.2)] that $q = wR^*$. Now by "Going-up" in the integral extension $\tilde{R} \to \tilde{R} \otimes_R T = \tilde{T}$, there is a minimal prime $\tilde{w}$ in $\tilde{T}$ and a (unique) maximal ideal Q in T (so $Q\tilde{T}$ is a maximal ideal in $\tilde{T}$) with $\tilde{w} \subseteq Q\tilde{T}$, $\tilde{w} \cap \tilde{R} = w$ and $Q\tilde{T} \cap \tilde{R} = P\tilde{R}$ (the maximal ideal of $\tilde{R}$). Now $T_Q^* = \tilde{T}_Q^*$, so there is a minimal prime v in T_Q^* such that $v \cap \tilde{T}_Q = \tilde{w}_Q$. It is easy to see that Q_Q^* is minimal over $IT_Q^* + v$, and the result follows.

Remark. Note that if T is a finite R-module, then $R_P^* \otimes_R T$ is the PT-adic completion of T and so is Noetherian.

Finally we extend McAdam's result on the contraction of quintasymptotic primes, using once again the ideas in the proof of Theorem 1.

THEOREM 2. *Suppose that every minimal prime in* T *contracts to a minimal prime in* R. *Let* $Q \in \bar{Q}^*(IT)$ *and let* $P = Q \cap R$.

(i) *If* $R_P^* \otimes_R T_Q$ *is Noetherian and excellent, then* $P \in \bar{Q}^*(I)$.

(ii) *If* T_Q *is excellent (and so if* T *is excellent), then* $P \in \bar{Q}^*(I)$.

Proof. (i) We assume that R is local at P. Since $Q \in \bar{Q}^*(IT)$, there exists a minimal prime w in T_Q^* such that Q_Q^* is minimal over $IT_Q^* + w$. By Going-down, $w \cap T'_Q$ is a minimal prime in T'_Q so that $w \cap T'$ is a minimal prime in T'. By the proposition, $w^* := w \cap R^* = w \cap T' \cap R^*$ is a minimal prime in R^*. Now QT' is the unique maximal prime ideal in T' which contains $w' := w \cap T'$. By hypothesis, T'_Q is excellent. Then T'_Q/w'_Q is excellent and Henselian (cf. [9, (43.4)], say). It follows from [3, (18.9.2)] that $w = w'_Q T^*_Q$, and it is then easy to see that P^* is minimal over $IR^* + w^*$, using Going-up in the integral extension $R^* \to T'$. The result follows.

(ii) As before, we may assume that R is local and that Q^*_Q is minimal over $IT^*_Q + y$ for some minimal prime y in T^*_Q. Let $\tilde{y} = y \cap \tilde{T}_Q$. By Going-down, $\tilde{y}$ is a minimal prime in $\tilde{T}_Q = \tilde{R} \otimes_R T_Q$. We now apply the proposition to the case of the (faithfully) flat extension $R \to \tilde{R}$: since $\tilde{y}$ corresponds under localization to a minimal prime of $\tilde{T} = \tilde{R} \otimes_R T$, it follows that $x := \tilde{y} \cap \tilde{R}$ is a minimal prime in $\tilde{R}$. As before, there is a minimal prime x^* in R^* lying over x. Again as before, $\tilde{T}_Q$ is an excellent ring and $\tilde{y} T^*_Q = y$. It is now clear from the Going-up property in the integral extension $\tilde{R} \subseteq \tilde{T}$ that P^* is minimal over $IR^* + x^*$. The result follows.

Remark. (1) If T is a finite R-module, then $R^*_P \otimes_R T$ is a finite R^*_P-module and so an excellent ring (since complete local rings are excellent). Hence, $R^*_P \otimes_R T_Q$ is an excellent ring.

(2) Since T is a direct limit of R-subalgebras T_α where each T_α is a finite R-module, the proof of [3, (18.7.5.1)] can easily be adapted (using the Going-up, "Incomparability," and base-change properties of integral extensions) to show that $R^* \otimes_R T$, if Noetherian, is certainly always universally catenary (cf. [5, p. 259]).

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